What can be verified locally?

Alkida Balliu

IRIF Paris and GSSI L'Aquila

Goal

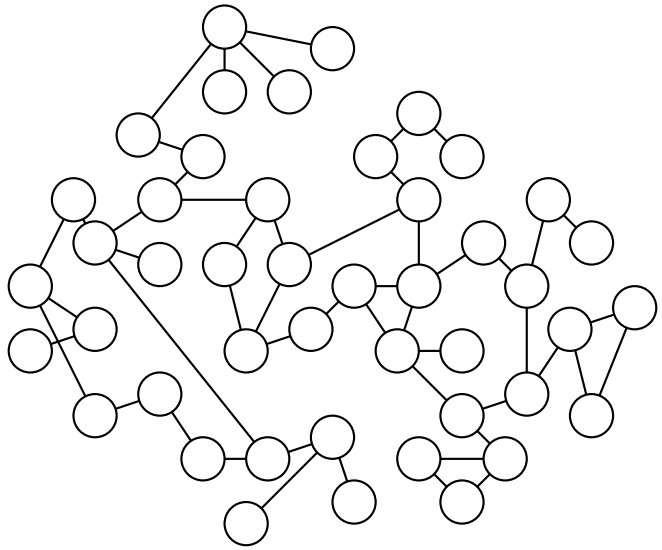
- Complexity theory in the distributed setting.
- Classify problems according to their difficulty.

Overview

- LOCAL model;
- Decision problems;
- Verification problems;
- Hierarchy of complexity classes in the LOCAL model.

LOCAL Model

- The distributed network is represented by a graph.



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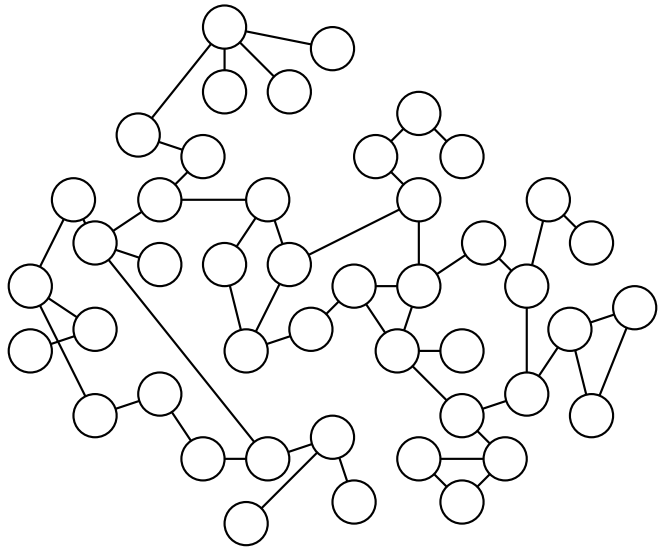

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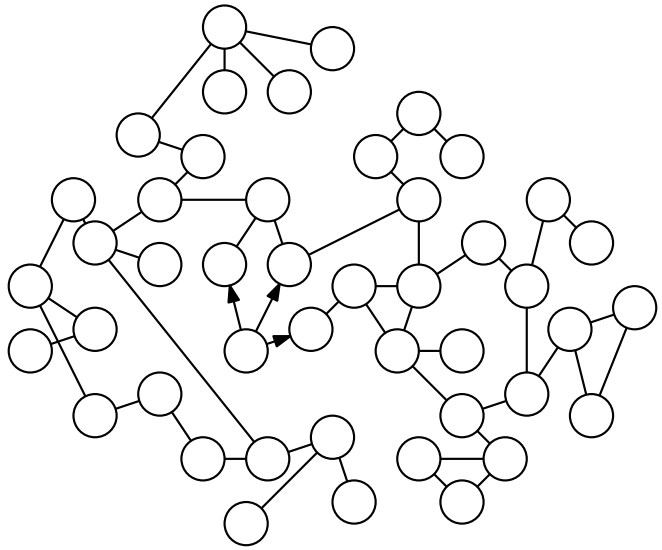

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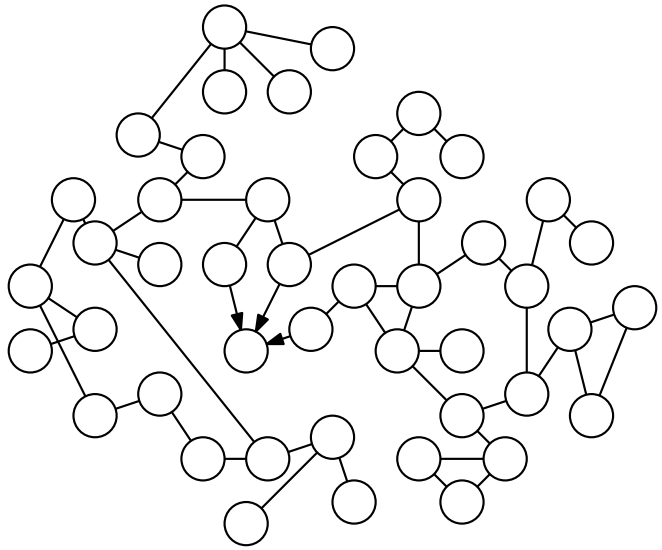

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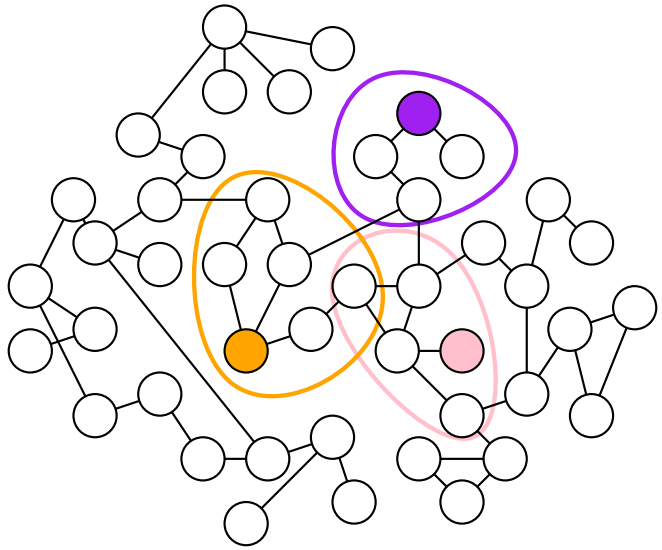

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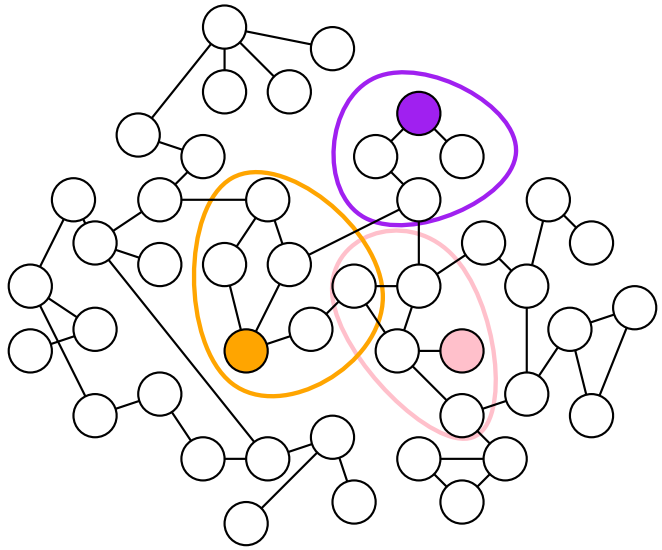

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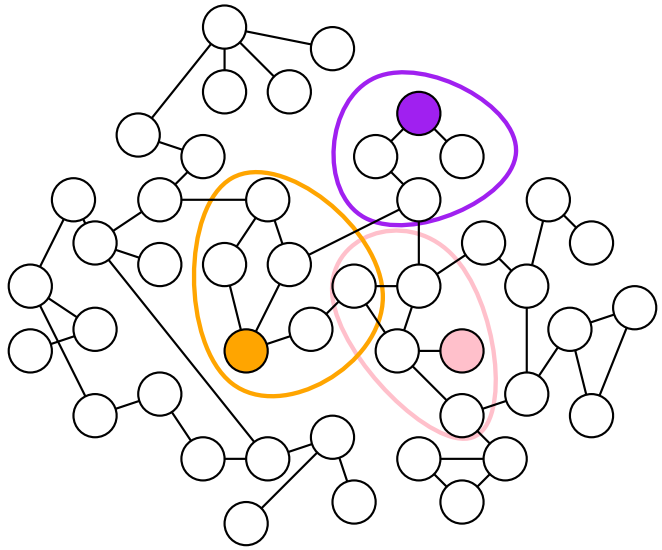
- We want t to be constant.



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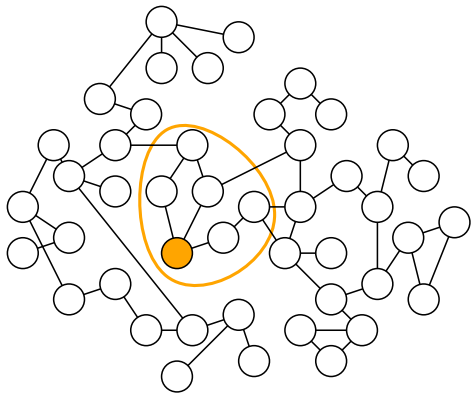

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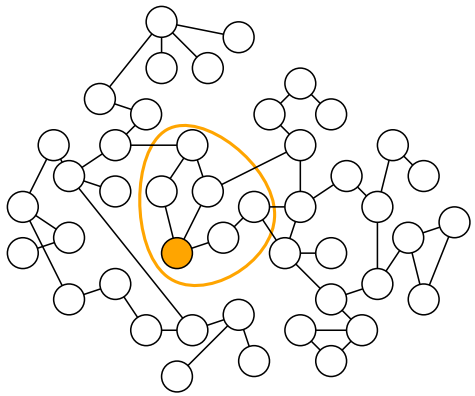

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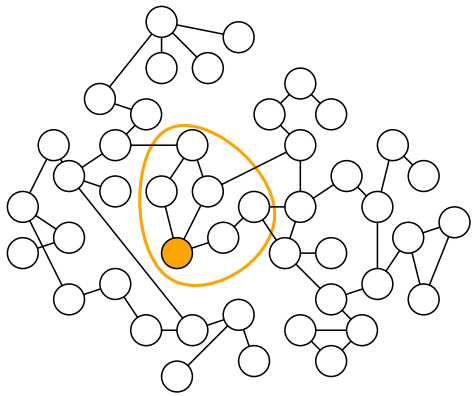

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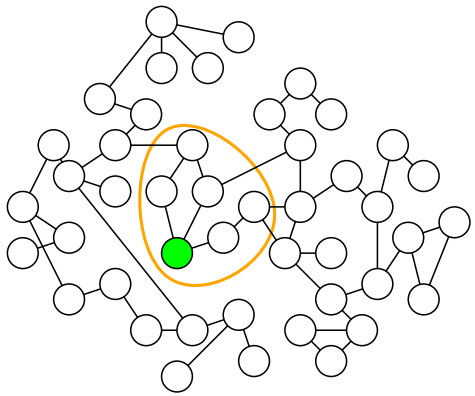

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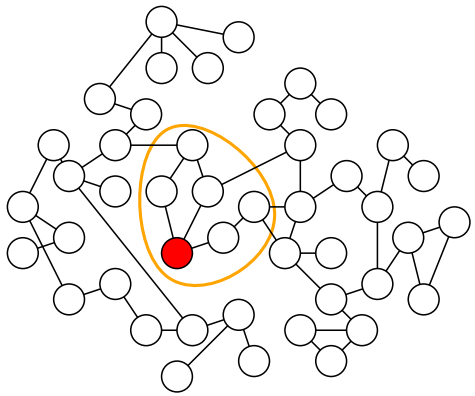

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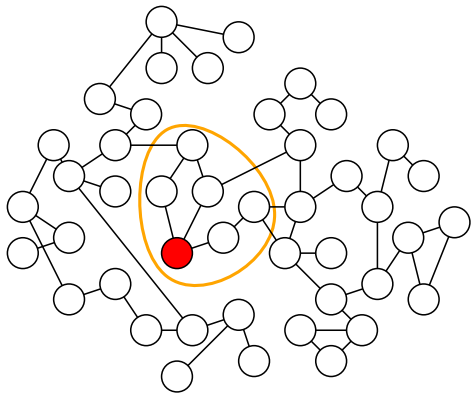

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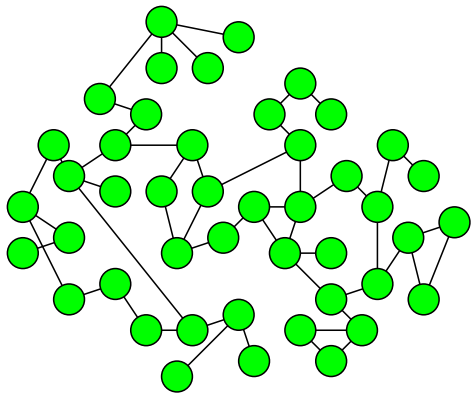

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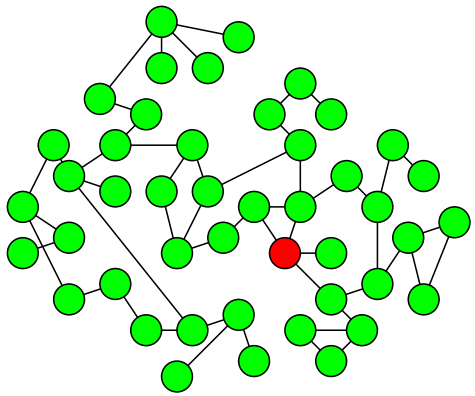

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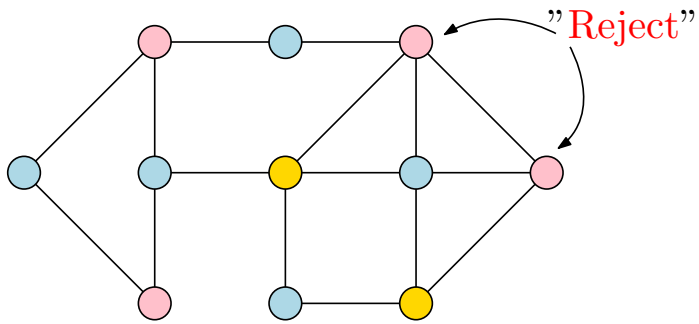

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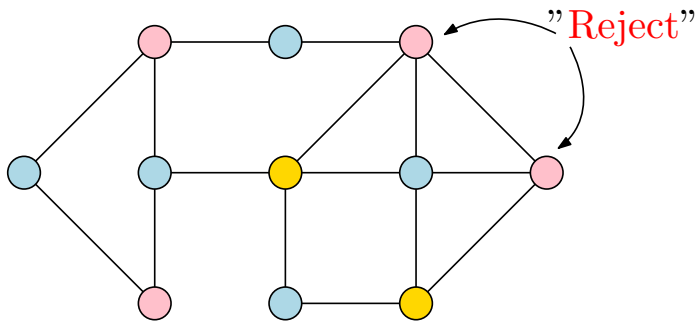

- *Local Decision (LD)* is the class of distributed languages that can be locally decided [NS '95].

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$\mathcal{L} \in \mathrm{P}$ if there is a polynomial time algorithm A such that,

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- Similar to PLS, but with id-independent certificates.

Example: Is the Given Graph a Tree?

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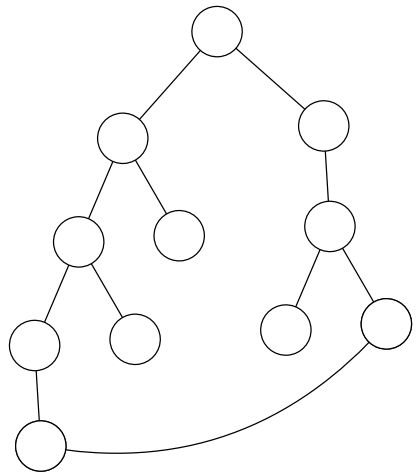

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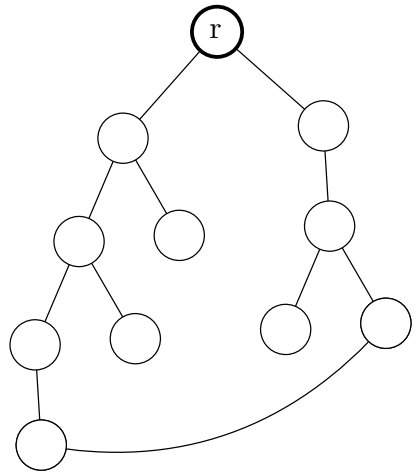

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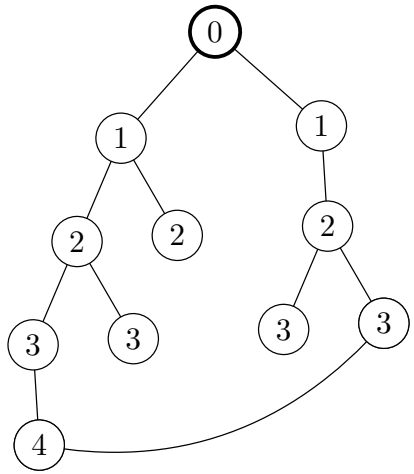

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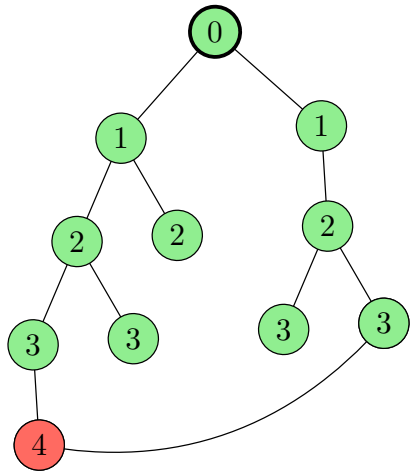

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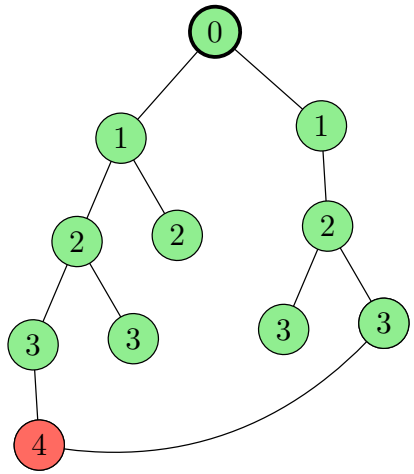

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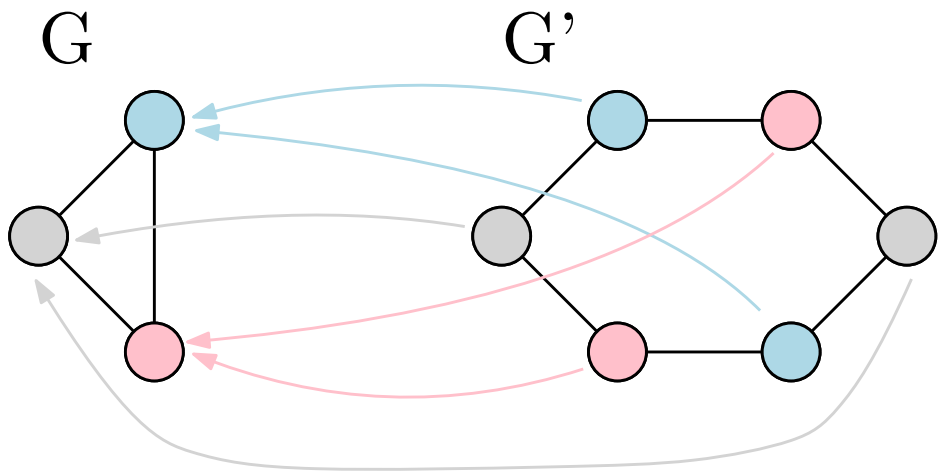
$\mathcal{L} \in$ NP if there is a polynomial time algorithm A such that,

$$x \in L \iff \exists c \text{ s.t. } A \text{ accepts } x \text{ with } c.$$

More About NLD

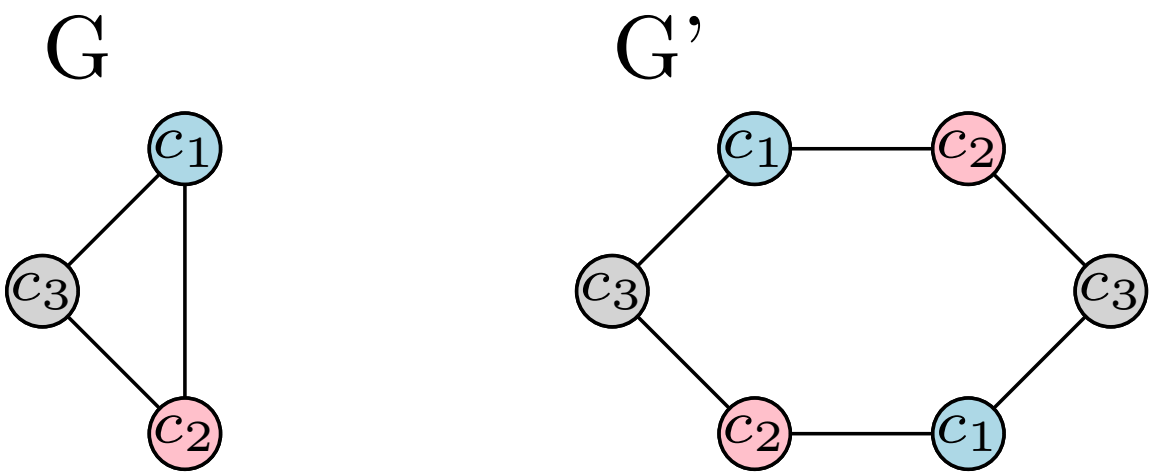
NLD is the class of all problems closed under lift [FKP '11].

- Let (G, x) and (G', x') be two input instances.
- (G', x') is a lift of (G, x) if there exists a function f such that: $f : V(G') \to V(G)$ preserving the local view of each node.



NLD is Closed Under Lift

- Let $\mathcal{L}$ be a language in NLD.
- If $(G, x) \in \mathcal{L} \ \wedge \ (G', x')$ is a lift of (G, x), then $(G', x') \in \mathcal{L}$.



Goal

- Build a hierarchy of complexity classes in the distributed setting.

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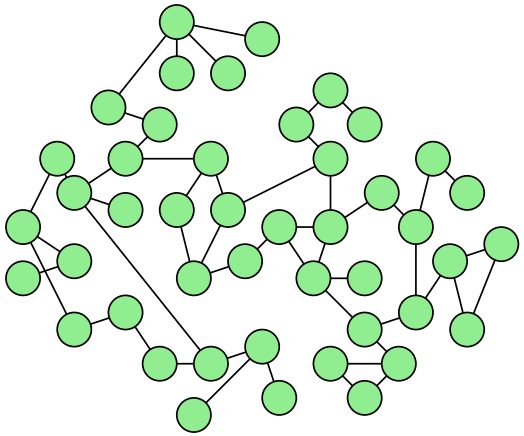
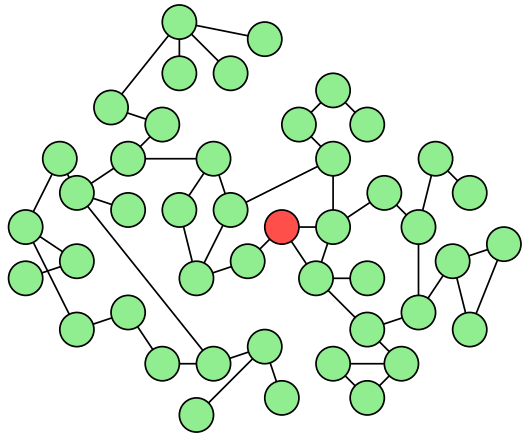
Complementary Classes

In a class:



A globaly accepted input instance.



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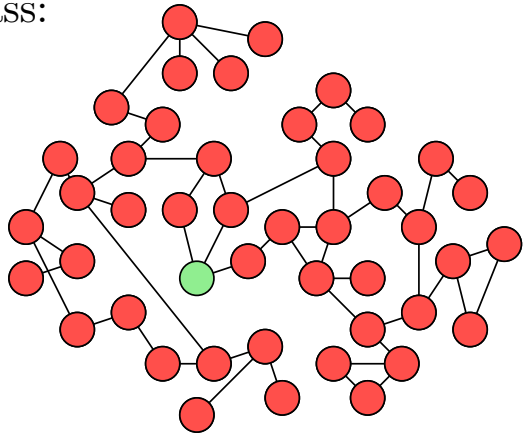

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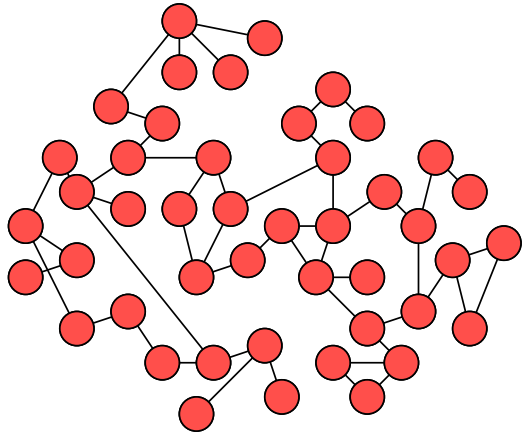

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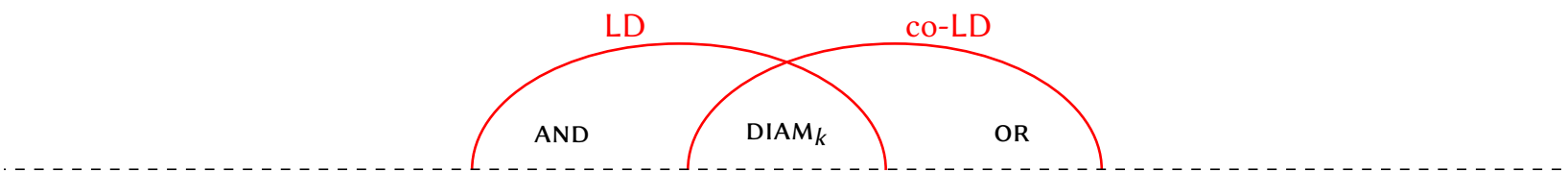
Level 0 of the Hierarchy

- AND : $\left|\{u \in V(G) : x(u) = 1\}\right| = 0$
- OR : $\left|\{u \in V(G) : x(u) = 1\}\right| \geq 1$



Π_1^{loc}: The Role of the Last Universal Quantifier

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- Problems that can be solved only if a specific node knows (an upper bound of) the size of the network!

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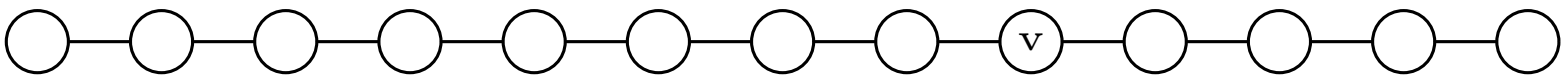

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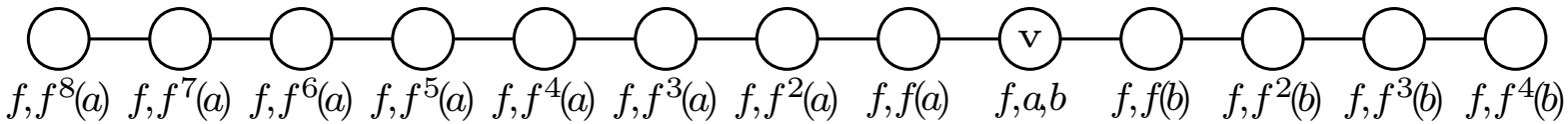

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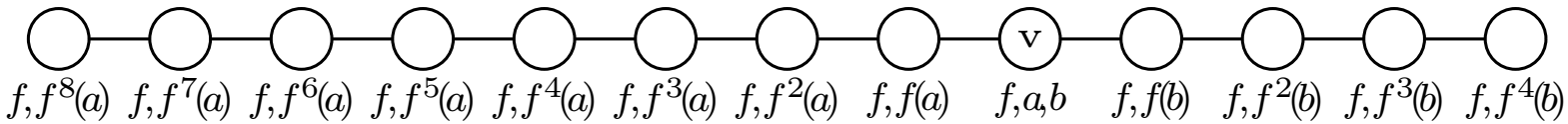

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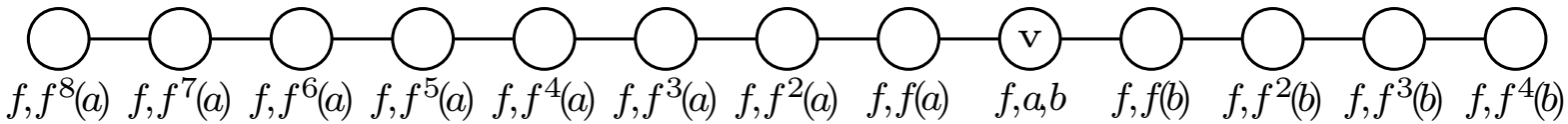
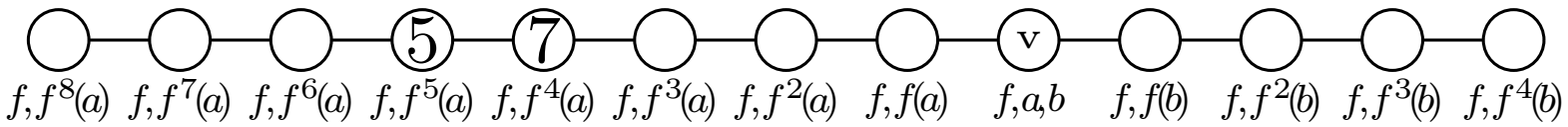
- Nodes in L (resp., in R) are given as input $f, f^i(a)$ (resp., $f, f^i(b)$); to v is given in input f, a, b.
- f is s.t. $f(0) = 0$ and $f(1) = 1$
- An input instance is in ITER if and only if:
  - $f^{|L|}(a) \in \{0, 1\}$ and $f^{|R|}(b) \in \{0, 1\}$
  - $f^{|L|}(a) = 0$ or $f^{|R|}(b) = 0$



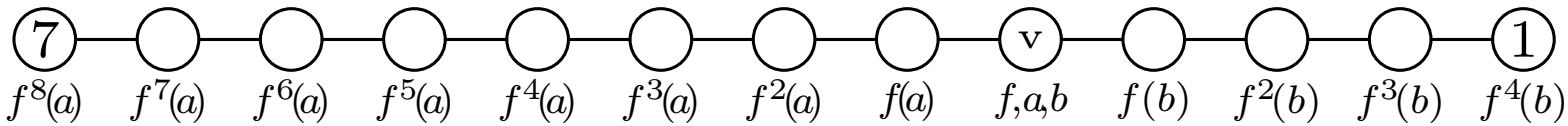
ITER



$$f(7) = 6 \neq 5$$

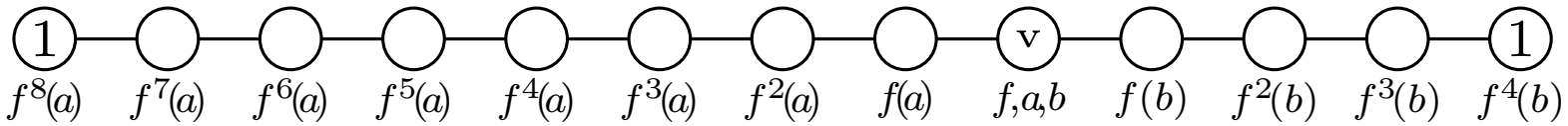
- Nodes reject if they notice local inconsistencies.

ITER



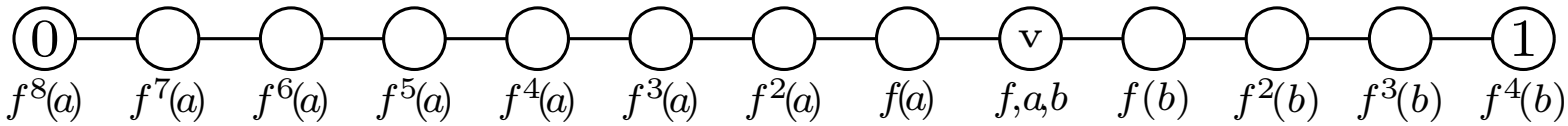
- An endpoint node rejects only if it has in input something different from 1 or 0; otherwise accepts.
- In this case, the left endpoint node rejects.

ITER



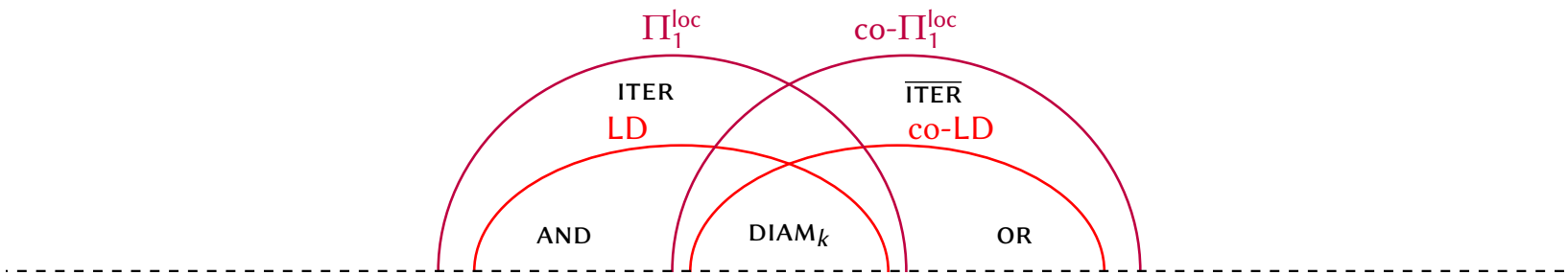
- $(G, x) \notin \mathcal{L} \Rightarrow \exists c$ s.t. at least one node rejects.
- v rejects only if $f^{|L|}(a) = f^{|R|}(b) = 1$; otherwise accepts.
- Certificate: an upper bound of the size of the network.

ITER

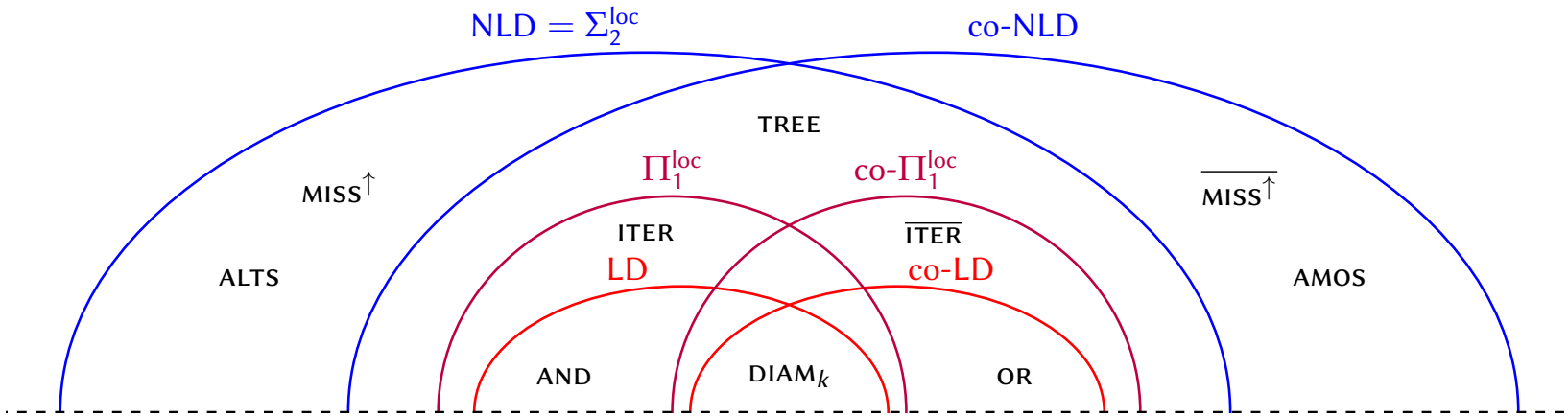


- $(G, x) \in \mathcal{L} \Rightarrow \forall c$ s.t. all nodes accept.
- Whatever certificate v has, it will never compute $f^{|L|}(a) = f^{|R|}(b) = 1$.

Local Hierarchy



Local Hierarchy



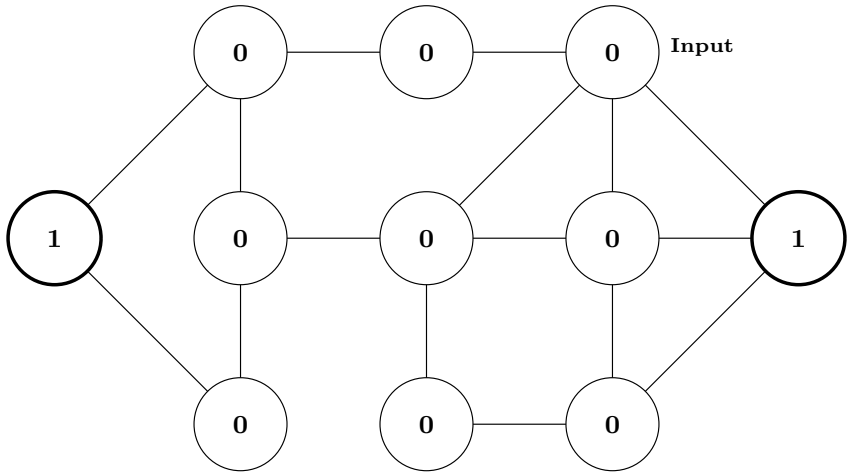
Π_2^{loc} Class

- Π_2^{loc} class: An input instance satisfies a certain property in Π_2 iff

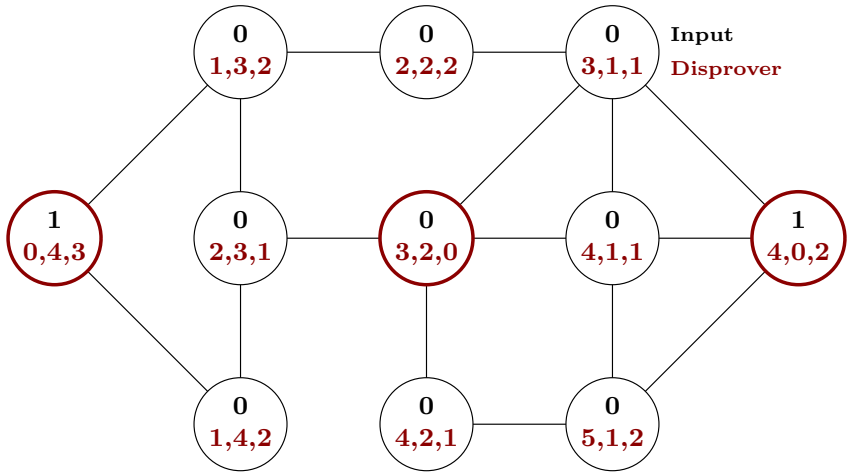
$$\forall c_1, \exists c_2, \text{ all nodes accept.}$$

- Two party game between a *disprover* and a *prover*.

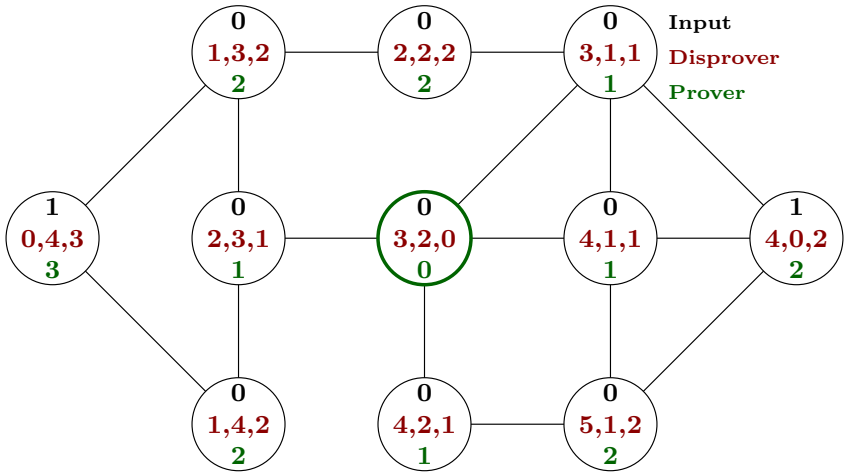
Exactly Two Selected



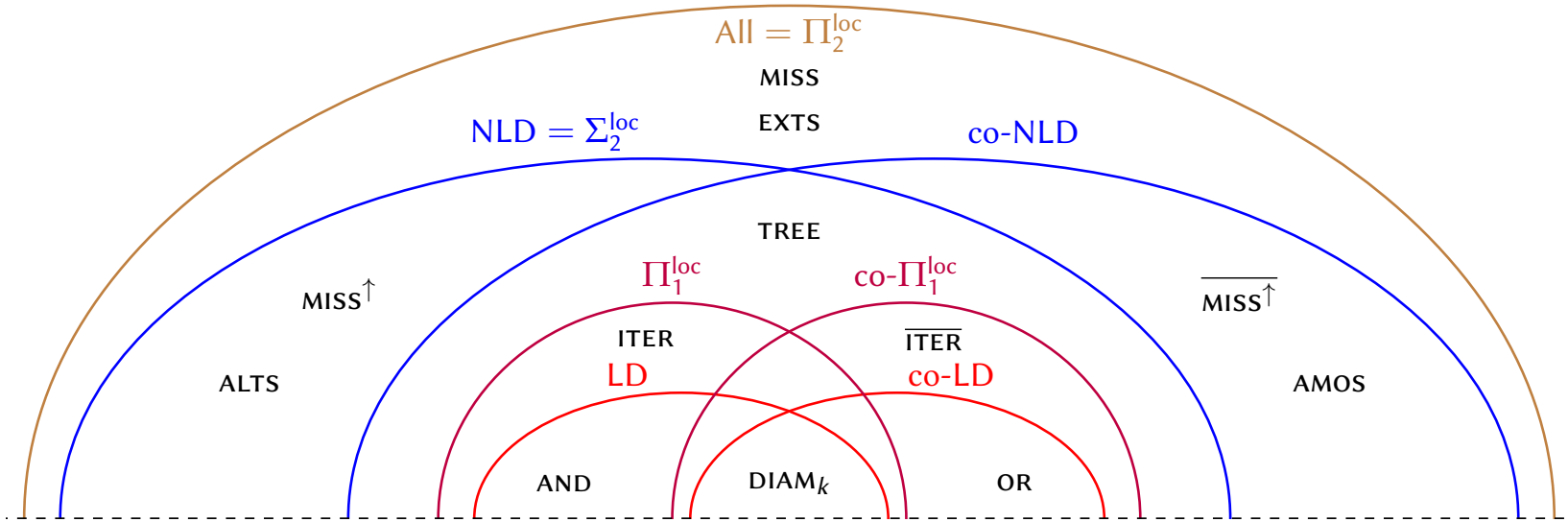
Exactly Two Selected



Exactly Two Selected



Local Hierarchy [B., D'Angelo, Fraigniaud, Olivetti '17]



$$\text{LD} \subset \Pi_1^{loc} \subset \text{NLD} = \Sigma_2^{loc} \subset \Pi_2^{loc} = \text{All (all inclusions are strict).}$$

Another local hierarchy [Feuilloley, Fraigniaud, Hirvonen '16]

The Context

- LOCAL model.
- Bounded-size certificates: $O(\log n)$ bits.
- Id-dependent certificates.

Another local hierarchy [Feuilloley, Fraigniaud, Hirvonen '16]

The Context

- LOCAL model.
- Bounded-size certificates: $O(\log n)$ bits.
- Id-dependent certificates.

- **The hierarchy**: $\forall i \geq 0$, λ_i is the class of distributed languages $\mathcal{L}$ s.t. $\exists A$, s.t. $\forall$ Id-assignment, the input instance is in $\mathcal{L}$ iff:

$$Qc_1 \ldots \forall c_{i-1} \exists c_i \text{ s.t. } \forall v \in V,\ A(G, x, c_1, \ldots, c_i, Id, v) = \text{accept}$$

where $Qc_1 = \forall$ if i is even and $Qc_1 = \exists$ otherwise.

The λ_i hierarchy

- $\lambda_0 = \text{LD}$.
- λ_1 similar to NLD.
- λ_2 similar to Π_2^{loc}.

  $\vdots$

- $\lambda_i = Qc_1 \ldots \forall c_{i-1} \exists c_i$ s.t. $\forall v \in V,\ A(G, x, c_1, \ldots, c_i, Id, v) = \text{accept}$

The λ_i hierarchy

- $\lambda_0 = \mathsf{LD}$.
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  $\vdots$
- $\lambda_i = Qc_1 \ldots \forall c_{i-1} \exists c_i$ s.t. $\forall v \in V,\ A(G, x, c_1, \ldots, c_i, Id, v) =$ accept
- Let λ_i^o be the corresponding class of λ_i where certificates are of $O(\log n)$ bits and id-oblivious.

Mapping between the 2 hierarchies

- $\lambda_1^o \subsetneq \lambda_1$
  Certifying a spanning tree: not closed under lift.

Mapping between the 2 hierarchies

- $\lambda_1^o \subsetneq \lambda_1$
  Certifying a spanning tree: not closed under lift.
- $\lambda_i^o = \lambda_i, \forall i \geq 2$
  Idea: give an Id assignment with the first certificate and let the other certificates to depend on this Id assignment (also check the correctness of this Id assignment).

Open Problems

- Unbounded size id-independent certificates:
  - find a complete problem for Π_1^{loc} and co-Π_1^{loc};
  - find a problem in the intersection between the classes Π_1^{loc} and co-Π_1^{loc}.
- Bounded size ($O(\log n)$) id-dependent certificates
  - we don't know if the hierarchy collapses;
  - there are no separating problems for λ_2 and λ_3 (neither for classes higher in the hierarchy).

Thank you!

References

- Alkida Balliu, Gianlorenzo D'Angelo, Pierre Fraigniaud, and Dennis Olivetti. What can be verified locally?, *STACS 2017*.
- Alkida Balliu, Gianlorenzo D'Angelo, Pierre Fraigniaud, and Dennis Olivetti. Brief Announcement: Local distributed verification, *DISC 2016*.
- Pierre Fraigniaud, Amos Korman, and David Peleg. Towards a complexity theory for local distributed computing, *J. ACM 2013 (FOCS 2011)*.
- Laurent Feuilloley, Pierre Fraigniaud, and Juho Hirvonen. A hierarchy of local decision, *ICALP 2016*.
- Fabian Reiter. Distributed graph automata, *LICS 2015*.
- Moni Naor and Larry Stockmeyer. What can be computed locally? *SIAM J. Comput. 1995*.